

# RATIONALITY AND THE JORDAN-GATTI-VINIBERGH DECOMPOSITION

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ABSTRACT. We verify the conjecture of [10] and use it to prove that the semisimple parts of the rational Jordan-Kac-Vinberg decompositions of a rational vector all lie in a single rational orbit.

## 1. INTRODUCTION

Let  $k$  be a field of characteristic 0, and write  $\bar{k}$  for its algebraic closure. Let  $G$  be a reductive algebraic group (not necessarily connected), acting on a vector space  $V$ , with  $G$ ,  $V$ , and the action all defined over  $k$ . Given a point  $v \in V$ , write  $G_v$  for the stabilizer of  $v$ ; it is an algebraic subgroup of  $G$ .

In [6], Kac and Vinberg made the following definitions:

- Definition 1.1.** (a) A vector  $s \in V$  is *semisimple* if the orbit  $G \cdot s$  is Zariski closed.  
 (b) A vector  $n \in V$  is *nilpotent* with respect to  $G$  if the Zariski closure  $\overline{G \cdot n}$  contains the vector 0.  
 (c) A *Jordan decomposition* of a vector  $\gamma \in V$  is a decomposition  $\gamma = s + n$ , with
- (1)  $s$  semisimple,
  - (2)  $n$  nilpotent with respect to  $G_s$ ,
  - (3)  $G_\gamma \subseteq G_s$ .

This Jordan-Kac-Vinberg decomposition matches the standard Jordan decomposition when  $V$  is the Lie algebra  $\mathfrak{g}$  of  $G$ . In that case, every  $\gamma \in \mathfrak{g}$  has a unique Jordan decomposition  $\gamma = s + n$ , and if  $\gamma$  lies in  $\mathfrak{g}(k)$  then so do  $s$  and  $n$ . For general  $V$ , however, as noted in [7], an element  $\gamma \in V$  may have multiple Jordan-Kac-Vinberg decompositions. For all of them, the element  $s$  lies in a single  $G$ -orbit, namely the unique closed  $G$ -orbit in  $\overline{G \cdot \gamma}$ .

In [7], Kac showed that a simple application of the Luna slice theorem shows that every vector  $\gamma \in V$  has a Jordan-Gatti-Vinberg decomposition (as he called it). A rational version of the Luna slice theorem has been proven by Bremigan [5], and it implies (Lemma 4.3) that every  $k$ -point  $\gamma \in V(k)$  has a  $k$ -Jordan-Kac-Vinberg decomposition, that is a Jordan-Kac-Vinberg decomposition  $\gamma = s + n$  with  $s$  (and hence  $n$ ) in  $V(k)$ . This fact has not previously appeared in the literature.

In this paper we show that given  $\gamma \in V(k)$ , the semisimple parts  $S$  of all  $k$ -Jordan-Kac-Vinberg decompositions  $\gamma = s + n$  of  $\gamma$  all lie in a single  $G(k)$ -orbit. In other words, even though  $k$ -Jordan-Kac-Vinberg decompositions are not unique, the  $G(k)$ -orbit of the semisimple parts is. This uniqueness is important in producing the fine geometric expansion in relative trace formulas (see the discussion in [9]).

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We use two tools to prove this. One is the rational version of the Hilbert-Mumford Theorem, as proven by Kempf [8] and Rousseau. The Hilbert-Mumford Theorem allows us to restate the problem in terms of limits of the form  $\lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$ , for  $k$ -cocharacters  $\lambda$  in  $G$ .

The other is a recent rationality result of Bate-Martin-Röhrle-Tange [2] on such limits.

In fact we prove the somewhat more general result, Theorem 3.4, that given  $\gamma \in V(k)$ , the limit points  $\lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$  that are semisimple, ranging over all cocharacters  $\lambda$  defined over  $k$ , all lie in a single  $G(k)$ -orbit. This solves the conjecture in [10].

## 2. PRELIMINARIES

We begin with some notation. Let  $k$  be a field (of any characteristic), and write  $\bar{k}$  for its algebraic closure. Let  $G$  be a reductive algebraic group (not necessarily connected) defined over  $k$ . Write  $X^*(G)$  for the group of characters  $\chi: G \rightarrow \text{GL}(1)$ , and  $X_*(G)$  for the set of cocharacters  $\lambda: \text{GL}(1) \rightarrow G$ . Similarly write  $X^*(G)_k$  (resp.  $X_*(G)_k$ ) for those characters (resp. cocharacters) defined over  $k$ . Define the map  $\langle \cdot, \cdot \rangle: X_*(G) \times X^*(G) \rightarrow \mathbb{Z}$  by requiring the identity  $\chi(\lambda(t)) = t^{\langle \lambda, \chi \rangle}$ . The group  $G$  acts naturally on  $X_*(G)$ :

$$(g \cdot \lambda)(t) = g\lambda(t)g^{-1}, \text{ for } g \in G, \lambda \in X_*(G), t \in \text{GL}(1).$$

Given  $\lambda \in X_*(G)_k$  and  $g \in G(k)$ , the cocharacter  $g \cdot \lambda$  is also in  $X_*(G)_k$ .

Suppose that  $V$  is an affine  $G$ -variety. Given  $\lambda \in X_*(G)$  and  $v \in V$ , we say that the *limit*

$$(2.1) \quad \lim_{t \rightarrow 0} \lambda(t) \cdot v$$

exists and equals  $x$  if there is a morphism of varieties  $\ell: \mathbb{A}^1 \rightarrow V$  with  $\ell(t) = \lambda(t) \cdot v$  for  $t \neq 0$ , and  $\ell(0) = x$ . Notice that if  $\ell$  exists then it is unique; also if  $V$  and  $\lambda$  are defined over  $k$ , with  $v \in V(k)$ , then  $\ell$  must also be defined over  $k$ , and so  $x$  must lie in  $V(k)$ . Given  $v \in V(k)$ , write  $\Lambda(v, k)$  for the set of  $\lambda \in X_*(G)_k$  such that the limit (2.1) exists.

The group  $G$  acts on itself via the action  $y \mapsto xyx^{-1}$ . Given  $\lambda \in X_*(G)$ , let  $P(\lambda)$  be the subvariety

$$P(\lambda) = \{ g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \}.$$

It is an algebraic group, defined over  $k$  if  $\lambda$  is. These groups  $P(\lambda)$  were defined in [11] and are called the Richardson parabolic subgroups in [2]. The map

$$h_\lambda: P(\lambda) \rightarrow G, \quad h_\lambda(g) = \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1}$$

is a homomorphism of algebraic groups, defined over  $k$  if  $\lambda$  is. The image and kernel are given by

$$\begin{aligned} \text{Im } h_\lambda &= G^\lambda = \{ g \in G \mid \lambda(t)g\lambda(t)^{-1} = g, \text{ for all } t \} \\ \ker h_\lambda &= \{ g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1 \} = R_u(P(\lambda)) \end{aligned}$$

(see [11] for details).

Now suppose that  $V$  is a  $G$ -module defined over  $k$ . Given any  $\lambda \in X_*(G)$ , we can then define the  $G^\lambda$ -modules

$$V_{\lambda,n} = \{v \in V \mid \lambda(t) \cdot v = t^n v \text{ for all } t\}, \quad n \in \mathbb{Z}$$

$$V_{\lambda,+} = \sum_{n>0} V_{\lambda,n}, \quad V_{\lambda,0+} = \sum_{n \geq 0} V_{\lambda,n} = V_{\lambda,0} \oplus V_{\lambda,+}.$$

Notice that  $V_{\lambda,0+}$  consists of those vectors  $v \in V$  such that the limit (2.1) exists, and is invariant under  $P(\lambda)$ ; in fact for  $g \in P(\lambda)$ , and  $v \in V_{\lambda,0+}$ ,

$$(2.2) \quad \lim_{t \rightarrow 0} \lambda(t) \cdot (g \cdot v) = h_\lambda(g) \cdot \left( \lim_{t \rightarrow 0} \lambda(t) \cdot v \right).$$

Further, for  $v \in V_{\lambda,0+} = V_{\lambda,0} \oplus V_{\lambda,+}$ , the limit (2.1) is just the projection of  $v$  to  $V_{\lambda,0}$ .

Suppose next that  $A$  is a maximal  $k$ -split torus in  $G$ . For each  $\chi \in X^*(A)_k$  define  $V^\chi$  by

$$V^\chi = \{v \in V \mid a \cdot v = \chi(a)v, \text{ for all } a \in A\}.$$

Then only finitely many  $V^\chi$  are nonzero and  $V$  is their direct sum. Given a vector  $v \in V$ , write  $v_\chi$  for the component of  $v$  in the space  $V^\chi$ ,  $\chi \in X^*(A)_k$ , and set

$$\text{supp } v = \text{supp}_A v = \{\chi \in X^*(A)_k \mid v_\chi \neq 0\},$$

so that

$$v = \sum_{\chi \in \text{supp } v} v_\chi.$$

For any  $\lambda \in X_*(A)_k \subset X_*(G)_k$ , each vector space  $V_{\lambda,n}$ ,  $n \in \mathbb{Z}$ , is also a direct sum of weight spaces:

$$V_{\lambda,n} = \sum_{\substack{\chi \in X^*(A)_k \\ \langle \lambda, \chi \rangle = n}} V^\chi.$$

We record the following obvious statement for later use.

**Lemma 2.3.** *For a vector  $v \in V$ , the limit (2.1) exists if and only if for every  $\chi \in \text{supp } v$  we have  $\langle \lambda, \chi \rangle \geq 0$ ; in this case the limit equals*

$$\sum_{\substack{\chi \in X^*(A)_k \\ \langle \lambda, \chi \rangle = 0}} v_\chi.$$

### 3. LIMITS

In this section we assume that  $k$  is perfect. We summarize some results that we will later use. First is the rational version of the Hilbert-Mumford Theorem, [8, Cor. 4.3].

**Lemma 3.1.** *If  $\gamma \in V(k)$ , then there exists  $\lambda \in X_*(G)_k$  so that the limit*

$$\lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$$

*exists and is semisimple.*

*Remark.* Note that this limit point must necessarily lie in  $V(k)$ .

The following two results are also essential to our proof. The first is a restatement of [2, Lemma 2.15].

**Lemma 3.2.** *Suppose that  $A \subset G$  is a torus and  $\lambda, \lambda_0$  are in  $X_*(A)_k$ . Suppose that vectors  $\gamma, v_0, v' \in V$  are related by*

$$\begin{aligned} v_0 &= \lim_{t \rightarrow 0} \lambda_0(t) \cdot \gamma, \\ v' &= \lim_{t \rightarrow 0} \lambda(t) \cdot v_0. \end{aligned}$$

*Then there exists  $\mu \in X_*(A)_k$  such that*

$$\begin{aligned} V_{\mu,0} &= V_{\lambda_0,0} \cap V_{\lambda,0} \\ V_{\mu,+} &\supseteq V_{\lambda_0,+}, \quad V_{\mu,0+} \subseteq V_{\lambda_0,0+}, \\ v' &= \lim_{t \rightarrow 0} \mu(t) \cdot \gamma. \end{aligned}$$

*Remark.* The cocharacter  $\mu$  can be of the form  $n\lambda_0 + \lambda$  for any sufficiently large  $n \in \mathbb{N}$ .

The second result is [2, Cor. 3.7].

**Lemma 3.3.** *Let  $v \in V(k)$  be semisimple. For every  $\lambda \in X_*(G)_k$ , if the limit  $\lim_{t \rightarrow 0} \lambda(t) \cdot v$  exists, then it lies in  $G(k) \cdot v$ .*

*Remark.* In fact [2, Cor. 3.7] shows that the limit must lie in  $R_u(P(\lambda))(k) \cdot v$ .

Our main result in this section is the following:

**Theorem 3.4.** *Let  $G$  be a reductive group and  $V$  a  $G$ -module. Suppose that  $k$  is perfect and let  $\gamma \in V(k)$ . Then for every  $\lambda, \mu \in X_*(G)_k$  such that both vectors  $v = \lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$  and  $v' = \lim_{t \rightarrow 0} \mu(t) \cdot \gamma$  exist and are semisimple,  $v'$  lies in  $G(k) \cdot v$ .*

*Remarks.* (a) This solves Conjecture 1.5 of [10].

(b) As is well-known (see for example [2, Remark 2.8] or [8, Lemma 1.1]), we can embed any affine  $G$ -variety over  $k$  inside a  $k$ -defined rational  $G$ -module, and hence Theorem 3.4 is also valid for affine  $G$ -varieties.

**Definition 3.5.** Let  $\Lambda(\gamma, k)_{\min}$  be the set of cocharacters that minimize  $\dim V_{\lambda,0}$ , among  $\lambda \in X_*(G)_k$  such that  $\lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$  exists and is semisimple.

*Remark.* By the Kempf-Rousseau-Hilbert-Mumford theorem 3.1, and because  $\dim V_{\lambda,0}$  is always a nonnegative integer, the set  $\Lambda(\gamma, k)_{\min}$  is non-empty.

**Lemma 3.6.** *Given  $\lambda \in \Lambda(\gamma, k)_{\min}$  and  $p \in P(\lambda)(k)$ , we have that  $\lambda \in \Lambda(p \cdot \gamma, k)_{\min}$ . Further, the limit points*

$$v = \lim_{t \rightarrow 0} \lambda(t) \cdot \gamma \quad \text{and} \quad \lim_{t \rightarrow 0} \lambda(t) \cdot (p \cdot \gamma)$$

*lie in the same  $G(k)$ -orbit.*

*Proof.* Let  $\lambda \in \Lambda(\gamma, k)_{\min}$ . By (2.2),  $\lim_{t \rightarrow 0} \lambda(t) \cdot (p \cdot \gamma) = h_\lambda(p) \cdot v$ , so the limit exists and lies in the  $G(k)$ -orbit of  $v$ ; consequently its  $G$ -orbit is closed.

On the other hand, given any  $\mu \in \Lambda(p \cdot \gamma, k)$ , we have that  $p^{-1} \cdot \mu \in \Lambda(\gamma, k)$  and  $\dim V_{\mu,0} = \dim V_{p^{-1} \cdot \mu,0}$ ; since  $\lambda \in \Lambda(\gamma, k)_{\min}$ , this dimension is at least  $\dim V_{\lambda,0}$ ; hence  $\lambda$  lies in  $\Lambda(p \cdot \gamma, k)_{\min}$ .  $\square$

**Lemma 3.7.** *Given  $\lambda_0 \in \Lambda(\gamma, k)_{\min}$ , write  $v_0 = \lim_{t \rightarrow 0} \lambda_0(t) \cdot \gamma$ . Suppose that  $A \subset G$  is a torus with  $\lambda_0 \in X_*(A)_k$ . Suppose that for  $\lambda \in X_*(A)_k$ , the limit  $v = \lim_{t \rightarrow 0} \lambda(t) \cdot \gamma$  exists and has a closed  $G$ -orbit. Then  $v$  lies in  $G(k) \cdot v_0$ .*

*Proof.* By Lemma 2.3 the existence of the limit  $v$  implies that for every  $\chi \in \text{supp}(\gamma)$  we have  $\langle \lambda, \chi \rangle \geq 0$ , and the vector  $v$  is the sum

$$\sum_{\substack{\chi \in \text{supp } \gamma \\ \langle \lambda, \chi \rangle = 0}} \gamma_\chi,$$

the projection of  $\gamma$  to  $V_{\lambda,0}$ ; in particular  $\text{supp } v \subseteq \text{supp } \gamma$  and  $\gamma - v \in V_{\lambda,+}$ . Similarly  $\text{supp}(v_0)$  is contained in  $\text{supp}(\gamma)$ , and so by Lemma 2.3 we may conclude that the limit

$$v' = \lim_{t \rightarrow 0} \lambda(t) \cdot v_0$$

exists. Since  $G \cdot v_0$  is closed,  $v'$  lies in  $G \cdot v_0$ , so that  $G \cdot v'$  is also closed.

We then obtain, from Lemma 3.2, a  $\mu \in \Lambda(\gamma, k)$  with  $v' = \lim_{t \rightarrow 0} \mu(t) \cdot \gamma$ , having a closed  $G$ -orbit, and

$$(3.8) \quad V_{\mu,0} = V_{\lambda_0,0} \cap V_{\lambda,0}$$

$$(3.9) \quad V_{\mu,+} \supseteq V_{\lambda_0,+}, \quad V_{\mu,0+} \subseteq V_{\lambda_0,0+},$$

Since  $\lambda_0$  lies in  $\Lambda(\gamma, k)_{\min}$ , we may conclude that  $V_{\mu,0} = V_{\lambda_0,0}$ , and hence by (3.8), (3.9), also that  $V_{\mu,0+} = V_{\lambda_0,0+}$ . The limit point  $v'$  is the projection of  $\gamma$  to  $V_{\mu,0} = V_{\lambda_0,0}$ , hence  $v' = v_0$ .

Since  $\lim_{t \rightarrow 0} \mu(t) \cdot \gamma$  exists,  $\gamma$  and hence  $v$  lie in  $V_{\mu,0+}$ . Now, the projection of  $\gamma - v$  to

$$V_{\lambda,0} = \sum_{\substack{\chi \in X^*(A)_k \\ \langle \lambda, \chi \rangle = 0}} V^\chi$$

is zero. By (3.8),  $V_{\mu,0} \subseteq V_{\lambda,0}$ , so the projection of  $\gamma - v \in V_{\mu,0+}$  to  $V_{\mu,0}$  is also zero, and hence

$$\lim_{t \rightarrow 0} \mu(t) \cdot v = \lim_{t \rightarrow 0} \mu(t) \cdot \gamma = v' = v_0.$$

By 3.3, we can finally conclude that  $v$  lies in  $G(k) \cdot v_0$ .  $\square$

*Proof of Theorem 3.4.* First, note that a cocharacter in  $G$  is necessarily a cocharacter in the connected component  $G^0$  of the identity in  $G$ , and that it is sufficient to prove Theorem 3.4 for  $G^0$ . Without loss of generality, we therefore assume that  $G$  is connected.

Pick  $\lambda_0 \in \Lambda(\gamma, k)_{\min}$ , set  $v_0 = \lim_{t \rightarrow 0} \lambda_0(t) \cdot \gamma$ . Since being in the same  $G(k)$ -orbit is an equivalence relation, it is clearly sufficient to prove the theorem for  $\mu = \lambda_0$ ,  $v' = v_0$ .

The image of  $\lambda_0$  lies in a maximal torus, and by [4, 1.4] must in fact lie in a maximal  $k$ -split torus  $A$ . Fix a minimal  $k$ -defined parabolic subgroup  $P$  of  $G$ , with  $C_G(A) \subseteq P \subseteq P(\lambda_0)$ . The choice of  $P$  corresponds to a choice of basis  ${}_k\Delta$  of simple roots of  $G$  with respect to  $A$ .

The image of  $\lambda$  also lies in some maximal  $k$ -split torus, so since all maximal  $k$ -split tori are conjugate over  $G(k)$  ([3, Thm. 20.9(ii)]), there exists  $g \in G(k)$  so that the image of  $g \cdot \lambda$  lies in  $A$ . Multiplying  $g$  on the left by an element of  $N_G(A)(k)$  if necessary, we can arrange that  $\langle g \cdot \lambda, \alpha \rangle \geq 0$  for every  $\alpha \in {}_k\Delta$ , that is,  $P \subseteq P(g \cdot \lambda)$ . Let us write  $\lambda_A$  for  $g \cdot \lambda \in X_*(A)$ .

We now apply the Bruhat decomposition: write

$$g = pwu, \quad p \in P(k) \subseteq P(\lambda_A)(k), \quad w \in N_G(A), \quad u \in R_u(P)(k).$$

Then

$$(3.10) \quad v = \lim_{t \rightarrow 0} \lambda(t) \cdot \gamma = g^{-1} \cdot \lim_{t \rightarrow 0} \lambda_A(t) g \cdot \gamma$$

$$(3.11) \quad = g^{-1} \cdot \left[ \lim_{t \rightarrow 0} \lambda_A(t) p \lambda(t)^{-1} \right] \cdot \lim_{t \rightarrow 0} \lambda_A(t) w u \cdot \gamma$$

$$(3.12) \quad = g^{-1} h_{\lambda_A}(p) \cdot \lim_{t \rightarrow 0} \lambda_A(t) w u \cdot \gamma$$

$$(3.13) \quad = g^{-1} h_{\lambda_A}(p) w \cdot \lim_{t \rightarrow 0} (w^{-1} \cdot \lambda_A)(t) \cdot (u \cdot \gamma),$$

with  $gh_{\lambda_A}(p)w \in G(k)$ . Note that the existence of the first limit in (3.11) implies the existence of the second.

Now,  $u \in R_u(P)(k) \subseteq P(k) \subseteq P(\lambda_0)(k)$ , so by Lemma 3.6,  $\lambda_0 \in \Lambda(u \cdot \gamma, k)_{\min}$ . Notice also that  $\lambda_0$  and  $w \cdot \lambda_A$  both lie in  $X_*(A)_k$ . By Lemmas 3.7 and 3.6,

$$\lim_{t \rightarrow 0} (w \cdot \lambda_A) \cdot (u \cdot \gamma) \in G(k) \cdot \lim_{a \rightarrow 0} \lambda_0(t) \cdot (u \cdot \gamma) = G(k) \cdot v_0$$

so  $v$  is also in  $G(k) \cdot v_0$ .  $\square$

#### 4. APPLICATION TO JORDAN DECOMPOSITIONS

In this section we require  $k$  to have characteristic 0.

**Definition 4.1.** (a) A *Jordan-Kac-Vinberg decomposition* of a vector  $\gamma \in V$  is a decomposition  $\gamma = s + n$ , with

- (1)  $s$  semisimple,
- (2)  $n$  nilpotent with respect to  $G_s$ ,
- (3)  $G_\gamma \subseteq G_s$ .

(b) Given  $\gamma \in V(k)$ , a *k-Jordan-Kac-Vinberg decomposition* of  $\gamma$  is a Jordan-Kac-Vinberg decomposition  $\gamma = s + n$  with  $s$  (and hence  $n$ ) in  $V(k)$ .

Kac [7] used the Luna Slice theorem to prove that every vector has a Jordan-Kac-Vinberg decomposition. We now show that every vector in  $V(k)$  has a  $k$ -Jordan-Kac-Vinberg decomposition.

Bremigan proved a rational version of the Luna Slice Theorem in [5]. The following, [5, Cor. 3.4] is an immediate consequence of it.

**Lemma 4.2.** *Given  $v \in V$  semisimple, let  $F$  be the set of points  $x \in V$  with  $G \cdot v \subseteq \overline{G \cdot x}$ . Then there is a  $G$ -invariant retraction  $\psi: F \rightarrow G \cdot v$  that is defined over  $k$ .*

*Remark.* For fields of positive characteristic, the Luna Slice Theorem does not hold without additional assumptions. See [1] for further details.

**Corollary 4.3.** *Every  $\gamma \in V(k)$  has a  $k$ -Jordan-Kac-Vinberg decomposition.*

*Proof.* Let  $\gamma \in V(k)$ . By Lemma 3.1, there exists a semisimple  $v \in \overline{G \cdot \gamma} \cap V(k)$ . Lemma 4.2 provides a  $G$ -invariant map  $\psi$ , defined over  $k$ , from

$$F = \{ x \in V \mid \overline{G \cdot x} \supseteq G \cdot v \}$$

to  $G \cdot v$ . Setting  $s = \psi(\gamma)$ , we immediately see that  $s \in V(k)$ , that  $s$  is semisimple, and that  $G_\gamma \subseteq G_s$ . Furthermore, since  $\psi$  is  $G$ -invariant,  $\psi(G_s \cdot \gamma) = s$ , and hence  $\psi(\overline{G_s \cdot \gamma}) = s$ , so that the unique closed  $G_s$ -orbit in  $\overline{G_s \cdot \gamma}$  is  $s$ . Subtracting  $s$ , the unique closed  $G_s$ -orbit in  $\overline{G_s \cdot (\gamma - s)}$  is 0. Therefore  $\gamma = s + (\gamma - s)$  is a  $k$ -Jordan-Kac-Vinberg decomposition.  $\square$

We can use the Hilbert-Mumford theorem to provide an alternate description of a Jordan-Kac-Vinberg decomposition.

**Proposition 4.4.** *A decomposition  $\gamma = s + n$ , with  $s$  semisimple, and  $G_\gamma \subseteq G_s$ , is a Jordan-Kac-Vinberg decomposition if and only if there exists  $\lambda \in X_*(G_s)$  so that*

$$(4.5) \quad \lim_{t \rightarrow 0} \lambda(t) \cdot \gamma = s.$$

*If  $\gamma \in V(k)$ , then  $\gamma = s + n$  is a  $k$ -Jordan-Kac-Vinberg decomposition if and only if  $\lambda$  can be taken to be in  $X_*(G_s)_k$ .*

*Proof.* The first part of the Proposition is just the second part over  $\bar{k}$ , so we need only consider the second part.

Given a  $k$ -Jordan-Kac-Vinberg decomposition  $\gamma = s + n$ , we know that  $0 \in \overline{G_s \cdot n}$ . The Hilbert-Mumford Theorem (Lemma 3.1) provides a  $\lambda \in X_*(G_s)_k$  such that

$$(4.6) \quad \lim_{t \rightarrow 0} \lambda(t) \cdot n = 0.$$

However, since the image of  $\lambda$  is in  $G_s$ , we can add  $s$  and obtain (4.5).

In the other direction, given  $\lambda \in X_*(G_s)_k$ , subtracting  $s$  from (4.5) gives (4.6), implying that  $n$  is nilpotent with respect to  $G_s$ . Since  $\gamma$  and  $\lambda$  are defined over  $k$ , so are  $s$  and  $n$ , hence  $\gamma = s + n$  is a  $k$ -Jordan-Kac-Vinberg decomposition.  $\square$

From Proposition 4.4 and Theorem 3.4, we immediately obtain the following:

**Corollary 4.7.** *For any two  $k$ -Jordan-Kac-Vinberg decompositions  $\gamma = s + n$ ,  $\gamma = s' + n'$  of  $\gamma \in V(k)$ , we have  $s' \in G(k) \cdot s$ .*

This means that although a vector  $\gamma \in V(k)$  may have multiple  $k$ -Jordan-Kac-Vinberg decompositions, all such decompositions lie in a single  $G(k)$ -orbit.

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